Lattice Based Cryptography and Fully Homomorphic Encryption

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NUMS

Introduction to Cryptography

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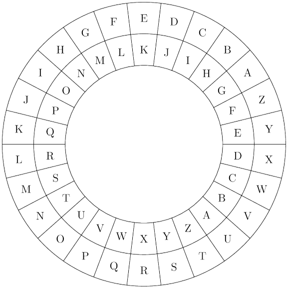

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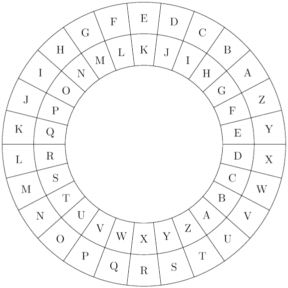

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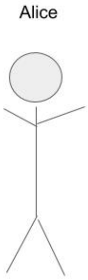
This scheme is super easy to break, so we needed something more

Public Key Cryptosystem



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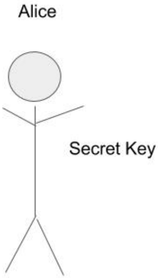

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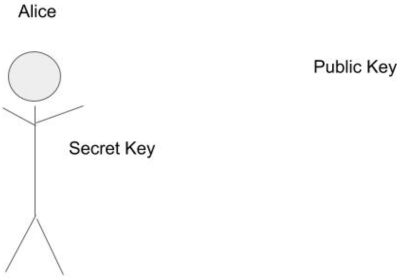

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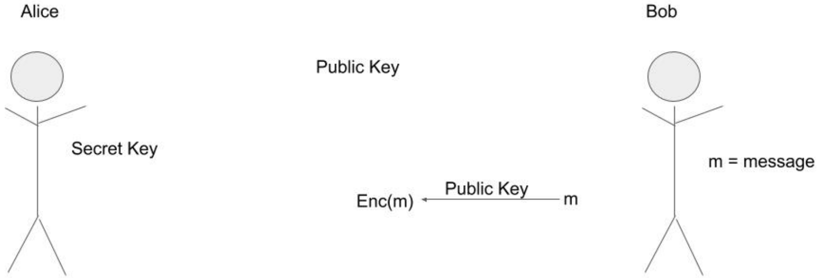

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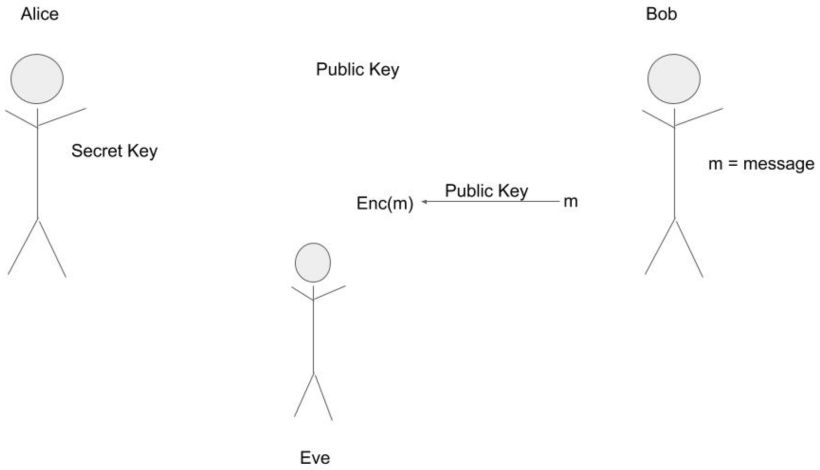

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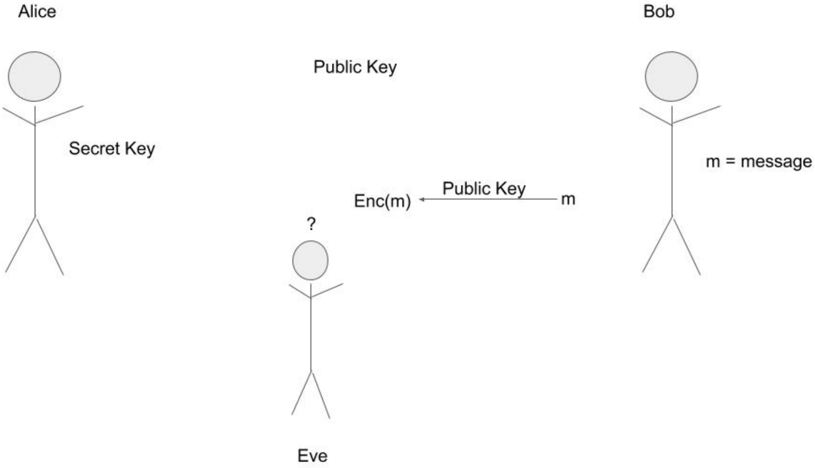

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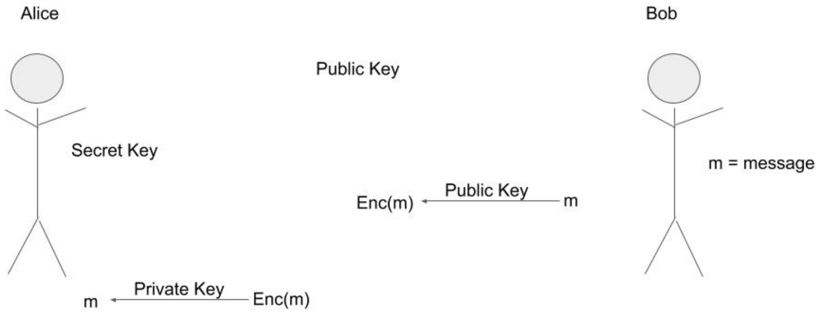

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RSA is based on the integer factoring problem being hard

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25
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35
75
15
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By adding a small amount of error a trivial problem becomes hard

Basic Scheme [BGV12]

Use the ring $R_q = \mathbb{Z}_q[x]/\langle x^d + 1 \rangle$
χ is the error distribution (over R_q)
$N = \lfloor \log q \rfloor$ number of samples for dRLWE to be well defined

Secret Key Generation:

pick $s' \leftarrow R_q$,
set SK: $\mathbf{s} = (1, s') \in R_q^2$

Public Key Generation:

pick $\mathbf{a}' \leftarrow R_q^N$ and $R_q^N \ni \mathbf{e} \leftarrow \chi^N$
$\mathbf{b} \leftarrow \mathbf{a}'s' + 2\mathbf{e}$.

$$\text{set PK: } \mathbf{A} = \begin{bmatrix} | & | \\ \mathbf{b} & -\mathbf{a}' \\ | & | \end{bmatrix} \in R_q^{N \times 2}$$

Note that $\mathbf{A} \cdot \mathbf{s} = 2\mathbf{e} \in R_q^N$

Basic Scheme Cont.

Encryption:

message $m \in R_2$, $\mathbf{m} = (m, 0) \in R_q^2$

$\mathbf{r} \leftarrow R_2^N$ a small random vector

ciphertext $\mathbf{c} = \mathbf{m} + \mathbf{A}^T\mathbf{r} = \begin{bmatrix} m \\ 0 \end{bmatrix} + \begin{bmatrix} \mathbf{b}^T\mathbf{r} \\ -\mathbf{a'}^T\mathbf{r} \end{bmatrix} \in R_q^2$

Decryption:

for a ciphertext $\mathbf{c}$ output $m \leftarrow [[\langle \mathbf{c}, \mathbf{s} \rangle]_q]_2$

$$\langle \mathbf{c}, \mathbf{s} \rangle = \langle \begin{bmatrix} (\mathbf{a'}^T s' + 2\mathbf{e}^T)\mathbf{r} + m \\ -\mathbf{a'}^T\mathbf{r} \end{bmatrix}, \begin{bmatrix} 1 \\ s' \end{bmatrix} \rangle = 2\mathbf{e}^T\mathbf{r} + m$$

As long as $\langle \mathbf{c}, \mathbf{s} \rangle < q/2$ then $[[\langle \mathbf{c}, \mathbf{s} \rangle]_q]_2 = [2\mathbf{e}^T\mathbf{r} + m]_2 = m$

$[x]_q$ denotes taking an $0 \le x \le q - 1$ to its representative in $(-q/2, q/2]$

Addition and Multiplication

For two ciphertexts $\mathbf{c}_1, \mathbf{c}_2$ encrypting messages m_1, m_2

Addition: $\mathbf{c}_1 + \mathbf{c}_2$ represents $m_1 + m_2$

$$\mathbf{c}_1 + \mathbf{c}_2 = \begin{bmatrix} m_1 + \mathbf{b}^T\mathbf{r_1} \\ -\mathbf{a'}^T\mathbf{r_1} \end{bmatrix} + \begin{bmatrix} m_2 + \mathbf{b}^T\mathbf{r_2} \\ -\mathbf{a'}^T\mathbf{r_2} \end{bmatrix} = \begin{bmatrix} m_2 + m_1 + \mathbf{b}^T(\mathbf{r_1} + \mathbf{r_2}) \\ -\mathbf{a'}^T(\mathbf{r_1} + \mathbf{r_2}) \end{bmatrix}$$

$$\langle (\mathbf{c}_1 + \mathbf{c}_2), \mathbf{s} \rangle = 2\mathbf{e}^T(\mathbf{r}_1 + \mathbf{r}_2)$$

Multiplication: $\mathbf{c}_1 \otimes \mathbf{c}_2$ encrypts $m_1 \cdot m_2$ under the *new* key $\mathbf{s} \otimes \mathbf{s}$

$$m_1 \cdot m_2 = [[\langle \mathbf{c}_1 \otimes \mathbf{c}_2, \mathbf{s} \otimes \mathbf{s} \rangle]_q]_2$$

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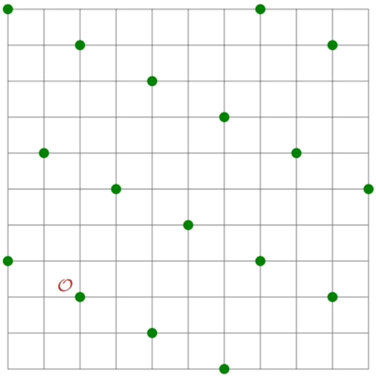
Also, how do we show that LWE problem is hard?

Lattice Problems

What is a lattice?

- A discrete additive subgroup of $\mathbb{R}^n$
- All linear combinations of some basis vectors

Lattices can exist in any dimension



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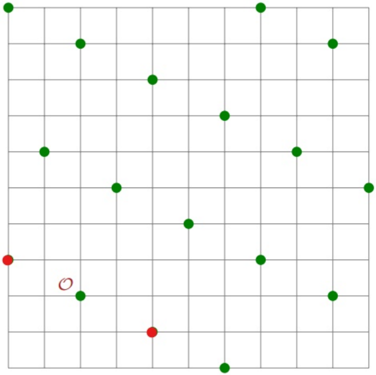

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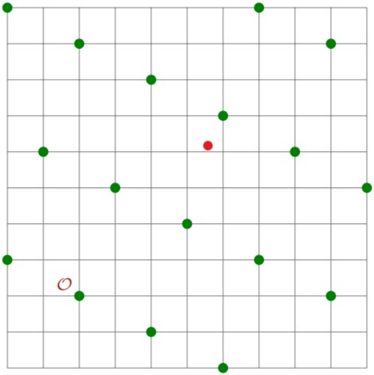

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We also have average case worst case reductions

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- We applied existing "noise management" techniques that do not compromise security
- When adding information that did not need to be encrypted, we found a way to incorporate unencrypted information with the encrypted information

(Ring) LWE Works Cited

1. **Regular LWE:**
   [Reg05] O. Regev. *On lattices, learning with errors, random linear codes, and cryptography*. In STOC, H. N. Gabow and R. Fagin, eds., ACM, New York, 2005, pp. 84–93.
2. **RLWE:**
   [LPR10] V. Lyubashevsky, C. Peikert, and O. Regev. *On ideal lattices and learning with errors over rings*. In EUROCRYPT, Springer, Berlin, 2010, pp. 1–23

Fully Homomorphic Encryption Schemes

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2. **RLWE Schemes**:

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   2. FHE Batching:

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